ST102 Week 16

Prop. Given i.i.d. sample $\{X_i\}_1^n$ from a smooth p.d.f. $f(x;\theta)$ with $\theta \in \mathbb{R}$. Denote MLE of θ by
$$\hat{\theta} = \hat{\theta}(X_1, \cdots, X_n).$$
Under some regularity requirements we have

$$\text{Distribution of } \sqrt{n}(\hat{\theta} - \theta) \xrightarrow{n \uparrow +\infty} N(0, \frac{1}{I(\theta)})$$

where $I(\theta)$ is the Fisher Information defined by

$$I(\theta) := -\int_{-\infty}^{+\infty} f(x;\theta) \frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2} dx$$

Rk. 1) When n is large, $\hat{\theta} \overset{\text{approx}}{\sim} N(\theta, \frac{1}{nI(\theta)})$

2) For discrete r.v. X, with mass function $p(x;\theta)$,

$$I(\theta) := -\sum_{x \in \mathcal{D}} p(x;\theta) \frac{\partial^2 \ln p(x;\theta)}{\partial \theta^2}.$$

<u>Construction of Confidence Intervals</u>

Motivation : coverage probability

$$\mathbb{P}(\mathcal{L}(\{X_i\}_1^n) < \theta < \mathcal{U}(\{X_i\}_1^n))$$

Case I: $\{X_i\}_1^n$ i.i.d. from $N(\mu, \sigma^2)$ with known σ^2.

By former lectures we know $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$.

$$\Rightarrow \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim N(0, 1)$$

Denote the critical value for 2-side tail prob. of α by $z_{\frac{\alpha}{2}}$, i.e. for $Y \sim N(0, 1)$,

$$\mathbb{P}(|Y| \geq z_{\frac{\alpha}{2}}) = \alpha.$$

Then we observe:

$$1 - \alpha = \mathbb{P}\left(\frac{|\bar{X} - \mu|}{\sigma / \sqrt{n}} < z_{\frac{\alpha}{2}} \right)$$

$$\Leftrightarrow \mathbb{P}\left(\bar{X} - z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}} \right) = 1 - \alpha$$

i.e. The $(1-\alpha)$ confidence interval for μ is

$$\left(\bar{X} - z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}} \right).$$

E.g. $\alpha = 0.05$, $1 - \alpha = 0.95$, $z_{\frac{\alpha}{2}} = 1.96$.

How to find $z_{\frac{\alpha}{2}}$? By symmetry.

Case II. $\{X_i\}_1^n$ i.i.d. from $N(\mu, \sigma^2)$, σ^2 unknown, still want inference on μ.

Recall $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$; $E.S.E.(\bar{X}) = \frac{S}{\sqrt{n}}$

It can be shown that $\frac{\bar{X} - \mu}{E.S.E.(\bar{X})} \sim t_{n-1}$

If we use $t_{n-1,\frac{\alpha}{2}}$ denote the critical value for 2-side tail prob. of t-distribution, then the $(1-\alpha)$ confidence interval for μ is

$$\left(\bar{X} - t_{n-1,\frac{\alpha}{2}} \cdot E.S.E(\bar{X}),\ \bar{X} + t_{n-1,\frac{\alpha}{2}} \cdot E.S.E.(\bar{X})\right)$$

Case III. $\{X_i\}_1^n$ i.i.d. from non-normal distribution with mean μ and variance σ^2.

The approximate $(1-\alpha)$ confidence interval is

$$\left(\bar{X} - z_{\frac{\alpha}{2}} \cdot \frac{S}{\sqrt{n}},\ \bar{X} + z_{\frac{\alpha}{2}} \cdot \frac{S}{\sqrt{n}}\right).$$