ST102 Week 18

Two Types of Errors

Truth \ Decision	H_0 not rejected	Reject H_0
H_0	Correct	Type I error
H_1	Type II error	Correct

H_0 v.s. H_1

Θ_0 $\quad$ Θ_1

Def. (Power function)

The power function of the test is

$$\beta(\theta) := \mathbb{P}_\theta(H_0 \text{ is rejected}), \text{ for } \theta \in \Theta_1.$$

Rk. For each $\theta \in \Theta_1$, we have $\beta(\theta) = 1 - \mathbb{P}(\text{Type II error})$

Tests for variances of Normals

Given i.i.d. random sample $\{X_i\}_1^n$ from $N(\mu, \sigma^2)$.

Test: $\quad H_0 : \sigma^2 = \sigma_0^2 \quad$ v.s. $\quad H_1 : \sigma^2 > \sigma_0^2$

Recall that $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$, then, under H_0,

$$T := \frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2_{n-1}$$

Reject H_0 if $T > \chi^2_{n-1,\alpha}$.

(continued)

For any given $\sigma^2 > \sigma_0^2$, i.e. any specific case in H_1, we can calculate the power of the test at σ by

$$\begin{aligned}
\beta(\sigma) &:= \mathbb{P}_\sigma(H_0 \text{ is rejected}) \\
&= \mathbb{P}_\sigma(T > \chi^2_{n-1,\alpha}) \\
&= \mathbb{P}_\sigma\left(\frac{(n-1)S^2}{\sigma_0^2} > \chi^2_{n-1,\alpha}\right) \\
&= \mathbb{P}_\sigma\left(\frac{(n-1)S^2}{\sigma_0^2}\frac{\sigma^2}{\sigma^2} > \chi^2_{n-1,\alpha}\right) \\
&= \mathbb{P}_\sigma\left(\frac{(n-1)S^2}{\sigma^2} > \frac{\sigma_0^2}{\sigma^2}\cdot\chi^2_{n-1,\alpha}\right) \\
&> \mathbb{P}_{\sigma_0}\left(\frac{(n-1)S^2}{\sigma_0^2} > \chi^2_{n-1,\alpha}\right) = \mathbb{P}(\text{Type I error})
\end{aligned}$$

Compare means of two Normals with paired data

$\{(X_i, Y_i)\}_1^n$, all independent, from $X_i \sim N(\mu_X, \sigma_X^2)$ and $Y_i \sim N(\mu_Y, \sigma_Y^2)$.

We want to test

$$H_0: \mu_X = \mu_Y \quad \text{v.s. some } H_1.$$

Trick: simplify the problem by letting $Z_i := X_i - Y_i$, $i \in [n]$.
Then we know $Z_i \sim N(\mu, \sigma^2)$ with $\mu = \mu_X - \mu_Y$ and $\sigma^2 = \sigma_X^2 + \sigma_Y^2$.

The primary problem can be equivalently formulated as

$$H_0: \mu = 0 \quad \text{v.s. some } H_1.$$

$\Rightarrow$ t-test.

(continued)

For a specific $\mu = \mu_X - \mu_Y > 0$ & one-sided test,

$$
\begin{aligned}
\beta(\mu) &= \mathbb{P}_\mu(H_0 \text{ is rejected}) \\
&= \mathbb{P}_\mu(T > t_{n-1,\alpha}) \qquad T := \frac{\bar{Z} - 0}{S/\sqrt{n}} \\
&= \mathbb{P}_\mu\left(\frac{\sqrt{n}\,\bar{Z}}{S} > t_{n-1,\alpha}\right) \\
&= \mathbb{P}_\mu\left(\frac{\bar{Z} - \mu + \mu}{S/\sqrt{n}} > t_{n-1,\alpha}\right) \\
&= \mathbb{P}_\mu\left(\underbrace{\frac{\bar{Z} - \mu}{S/\sqrt{n}}}_{\text{follow } t_{n-1}} > t_{n-1,\alpha} - \frac{\mu}{S/\sqrt{n}}\right) \\
&> \alpha \qquad (\text{and} \nearrow \text{ if } \mu \nearrow)
\end{aligned}
$$

<u>What if not paired</u>

Given independent i.i.d. random samples $\{X_i\}_1^n \sim N(\mu_X, \sigma_X^2)$ and $\{Y_i\}_1^m \sim N(\mu_Y, \sigma_Y^2)$. Test: $H_0: \mu_X = \mu_Y$,

Some facts:

1) $\bar{X}, \bar{Y}, S_X^2, S_Y^2$ are independent
2) $\bar{X} \sim N(\mu_X, \frac{\sigma_X^2}{n})$, $\frac{(n-1)S_X^2}{\sigma_X^2} \sim \chi^2_{n-1}$
3) $\bar{Y} \sim N(\mu_Y, \frac{\sigma_Y^2}{n})$, $\frac{(m-1)S_Y^2}{\sigma_Y^2} \sim \chi^2_{m-1}$
4) $\bar{X} - \bar{Y} \sim N(\mu_X - \mu_Y, \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{n})$

(continued)

We can then have the test statistic as

$$T := \frac{(\bar{X} - \bar{Y} - \mu_X + \mu_Y) \Big/ \sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}}{\sqrt{\left[\frac{(n-1)S_X^2}{\sigma_X^2} + \frac{(m-1)S_Y^2}{\sigma_Y^2}\right] \Big/ (n+m-2)}} \sim t_{n+m-2}$$

Case I. If $\sigma_X^2 = \sigma_Y^2$ but unknown

$$T = \sqrt{\frac{n+m-2}{\frac{1}{n} + \frac{1}{m}}} \cdot \frac{\bar{X} - \bar{Y} - (\mu_X - \mu_Y)}{\sqrt{(n-1)S_X^2 + (m-1)S_Y^2}} \sim t_{n+m-2}$$

Case II. σ_X^2 & σ_Y^2 known

$$T = \frac{\bar{X} - \bar{Y} - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \sim N(0, 1)$$

<u>Test for correlation coefficients</u>

Def. (Correlation Coefficients)

$$\rho = \mathrm{Corr}(X, Y) := \frac{\mathrm{Cov}(X, Y)}{[\mathrm{Var}(X)\,\mathrm{Var}(Y)]^{\frac{1}{2}}}$$

Def. (Sample correlation coefficients)

$$\hat{\rho} := \frac{\sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y})}{\left[\sum_{i=1}^{n} (X_i - \bar{X})^2 \sum_{j=1}^{n} (Y_j - \bar{Y})^2\right]^{\frac{1}{2}}}$$ for paired data $\{(X_i, Y_i)\}_1^n$

(continued)

Now we want to test

$$H_0: \rho = 0 \quad \text{v.s.} \quad \text{some } H_1$$

It can be proved that

$$T := \hat{\rho}\sqrt{\frac{n-2}{1-\hat{\rho}^2}} \sim t_{n-2}$$

Test for the ratio in variances of two Normals

Given i.i.d. random samples $\{X_i\}_1^n$ from $N(\mu_X, \sigma_X^2)$ and $\{Y_i\}_1^m$ from $N(\mu_Y, \sigma_Y^2)$. We want to test

$$H_0: \frac{\sigma_Y^2}{\sigma_X^2} = k \quad \text{v.s.} \quad H_1: \frac{\sigma_Y^2}{\sigma_X^2} \neq k \quad (\text{some } k > 0)$$

We then know

$$T := \frac{S_X^2 / \sigma_X^2}{S_Y^2 / \sigma_Y^2} \sim F_{n-1, m-1}$$